Exercise 2.10

Let $1 \le p \le q \le \infty$.

(a) Denote by ℓ_p^n the finite dimensional vector space $\mathbb{R}^n$ equipped with the norm $\|(x_1, \dots, x_n)\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$. Show that the norms $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent on $\mathbb{R}^n$

(b) Show that $\ell_p \subset \ell_q$, but ℓ_q is not a subset of ℓ_p.

Proof of (a): All the norms on a finite dimensional space are equivalent.

Proof of (b): (i) $q = \infty$.

- Pick any $x \in \ell_p$.

  Then $\sum_{k=1}^{\infty} |x_k|^p < \infty$

  $\Rightarrow |x_k|^p \to 0$ as $k \to \infty$

  $\Rightarrow |x_k|$ is bounded

  $\Rightarrow x \in \ell_\infty$

- Put $x_k = 1$. Then $x \in \ell_\infty$ but $x \notin \ell_p$ for any $p < \infty$.

(i) $q < \infty$

- Pick any $x \in \ell_p$.

  Then $\sum_{k=1}^{\infty} |x_k|^p < \infty$.

  Thus $|x_k|^p \to 0$ as $k \to \infty$.

  Then $\exists\ N \in \mathbb{N}$ such that $|x_k|^p \le 1$ for $k > N$.

  Thus $|x_k|^q \le |x_k|^p$ for $k > N$

Therefore, $\sum_{k=1}^{\infty}|x_k|^q = \sum_{k=1}^{N}|x_k|^q + \sum_{k>N}|x_k|^q$

$$\leq \sum_{k=1}^{N}|x_k|^q + \sum_{k>N}|x_k|^p$$

$$\leq \sum_{k=1}^{N}|x_k|^q + \sum_{k=1}^{\infty}|x_k|^p < \infty.$$

- Put $x_k = k^{-r}$ with $p < r < q$.

Then $\sum_{k=1}^{\infty} x_k^p = \sum_{k=1}^{\infty} k^{p-r} < \infty$

but $\sum_{k=1}^{\infty} x_k^q = \sum_{k=1}^{\infty} k^{q-r} = \infty$.

□

Exercise 3.4

Let X and Y be normed spaces. If x is a non-zero vector in X, and $y \in Y$, show that there exists a bounded linear map T such that $T(x) = y$.

Proof: Let $X' = \operatorname{span}\{x\}$ and $f(\alpha x) = \alpha$ for all $\alpha x \in X'$.

By Hahn-Banach Theorem, there exists a bounded linear function F on X such that $\|F\| = \|f\|$.

Define $T: X \to Y$ by $T(z) = F(z)y$ for all $z \in X$.

Then $T(x) = F(x)y = f(x)y = y$.

And $\|T(z)\| = \|F(z)\|\,\|y\| \leq \|F\|\,\|y\|\,\|z\| = \|f\|\,\|y\|\,\|z\|$

$$= \frac{\|y\|}{\|x\|}\|z\|.$$

□

Exercise 3.6

Let X, Y be normed spaces. Show that if $\mathcal{L}(X,Y)$ is Banach, then Y is Banach.

Proof: Pick any Cauchy sequence $(y_n) \in Y$.

Fix $x \in X$ with $\|x\| = 1$

By Ex 3.4, there exists $T_n \in \mathcal{L}(X,Y)$ such that

$T_n(x) = y_n$ and $\|T_n\| \leq \|y_n\|$.

Similarly, $\|T_n - T_m\| \leq \|y_n - y_m\|$.

Thus $(T_n) \in \mathcal{L}(X,Y)$ is a Cauchy sequence.

Since $\mathcal{L}(X,Y)$ is Banach, $\exists\ T \in \mathcal{L}(X,Y)$

such that $\|T_n - T\| \to 0$ as $n \to \infty$.

Put $y = T(x)$.

$$\begin{aligned} \text{Then } \|y_n - y\| &= \|T_n(x) - T(x)\| \\ &\leq \|T_n - T\| \|x\| \\ &= \|T_n - T\| \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence, Y is a Banach space.

$\square$